Phase Retrieval via Model-Free Power Flow Jacobian Recovery

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What is the problem?

Problem 1:
The case of the missing power network model

The case of the missing power network model

An electric power network is a graph

$$\mathcal{G} = (\mathcal{N}, \mathcal{V}).$$

The case of the missing power network model

Every node $i \in \mathcal{N}$ has a complex **voltage phasor** and **power injection**:

$$v_i \angle \theta_i \in \mathbb{C}, \qquad p_i + jq_i \in \mathbb{C}.$$

The case of the missing power network model

The network topology is encoded in the *admittance* matrix:

$$Y = G + jB \in \mathbb{C}^{n \times n}.$$

It is **hard to find** in the real world.

The case of the missing power network model

The non-linear power flow equations govern the power injections at every node $i \in \mathcal{N}$:

$$p_i = v_i \sum_{k=1}^{n} v_k (G_{ik} \cos \theta_{ik} + B_{ik} \sin \theta_{ik}), \tag{1a}$$

$$q_i = v_i \sum_{k=1}^{n} v_k (G_{ik} \sin \theta_{ik} - B_{ik} \cos \theta_{ik}). \tag{1b}$$

The case of the missing power network model

The non-linear power flow equations govern the power injections at every node $i \in \mathcal{N}$:

$$p_i = v_i \sum_{k=1}^{n} v_k (G_{ik} \cos \theta_{ik} + B_{ik} \sin \theta_{ik}), \tag{2a}$$

$$q_i = v_i \sum_{k=1}^{n} v_k (G_{ik} \sin \theta_{ik} - B_{ik} \cos \theta_{ik}). \tag{2b}$$

They are functions of the network topology.

Problem 2:
The case of the missing voltage phasor

The case of the missing voltage phasor

Every node $i \in \mathcal{N}$ has a complex **voltage phasor** and **power injection**:

$$v_i \angle \theta_i \in \mathbb{C}, \quad p_i + jq_i \in \mathbb{C}.$$

It's **hard to find** the voltage phase angles in the real world.

The case of the missing voltage phasor

Only $\approx$ 3000 phasor measurement units (PMUs) in North America [8].

Rare in:

1. Distribution systems
2. Transmission system boundaries
3. Rural transmission systems
4. Underserved areas

The case of the missing voltage phasor

Example: PJM, circa 2022

1. 400 PMUs throughout PJM territory[1]
2. Only required on substations

[1] PJM, "Synchrophasor Technology Roadmap", 2022.

The case of the missing voltage phasor

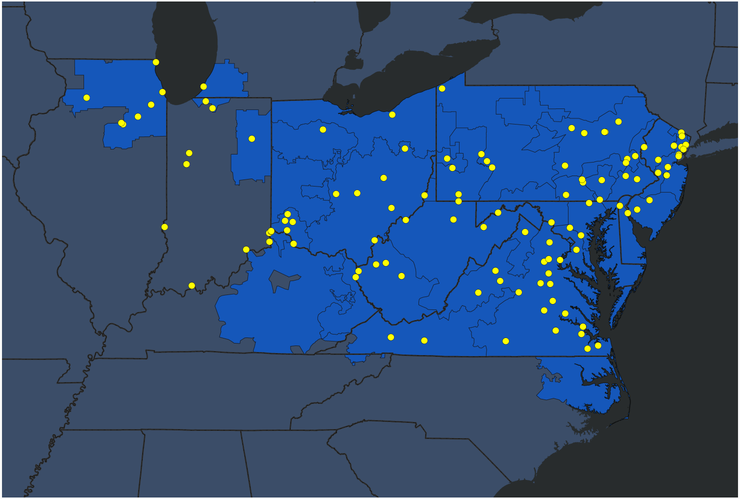

Figure 1: circa 2022 PMU deployment in PJM[2]

[2] Credit PJM, "Synchrophasor Technology Roadmap", 2022. Source: *https://www.pjm.com/markets-and-operations/ops-analysis/synchrophasor-technology*

OK, so who cares?

It’s pretty hard to be a power engineer without this information!

Case study: Newton-Raphson Power Flow model

How to solve the non-linear power flow equations?

Classic approach: *Newton-Raphson power flow.*

Iteratively solve a linear system of equations of the form

$$\begin{bmatrix} \Delta p \\ \Delta q \end{bmatrix} = \begin{bmatrix} \frac{\partial p}{\partial \boldsymbol{\theta}}(x) & \frac{\partial p}{\partial v}(x) \\ \frac{\partial q}{\partial \boldsymbol{\theta}}(x) & \frac{\partial q}{\partial v}(x) \end{bmatrix} \begin{bmatrix} \Delta \boldsymbol{\theta} \\ \Delta v \end{bmatrix} = J(x)\Delta x. \tag{3}$$

1. $\Delta \boldsymbol{p}, \Delta \boldsymbol{q} \in \mathbb{R}^n$ are small perturbations in the active and reactive power injections
2. $\Delta \boldsymbol{\theta}, \Delta v \in \mathbb{R}^n$ are small perturbations in the voltage phase angles and magnitudes

Case study: Newton-Raphson Power Flow model

$$\begin{bmatrix} \Delta p \\ \Delta q \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial p}{\partial \boldsymbol{\theta}}(x) & \frac{\partial p}{\partial v}(x) \\ \frac{\partial q}{\partial \boldsymbol{\theta}}(x) & \frac{\partial q}{\partial v}(x) \end{bmatrix}}_{:= J(x)} \begin{bmatrix} \Delta \boldsymbol{\theta} \\ \Delta v \end{bmatrix} = J(x)\Delta x, \tag{4}$$

1. The matrix $J(x) \in \mathbb{R}^{2n \times 2n}$ is the power flow Jacobian matrix.
2. Derivatives of the power flow equations (2) with respect to the voltage magnitudes $\boldsymbol{v}$ and phase angles $\boldsymbol{\theta}$.

Case study: Newton-Raphson Power Flow model

$$\begin{bmatrix} \Delta p \\ \Delta q \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial p}{\partial \boldsymbol{\theta}}(x) & \frac{\partial p}{\partial v}(x) \\ \frac{\partial q}{\partial \boldsymbol{\theta}}(x) & \frac{\partial q}{\partial v}(x) \end{bmatrix}}_{:=J(x)} \begin{bmatrix} \Delta \boldsymbol{\theta} \\ \Delta v \end{bmatrix} = J(x)\Delta x, \tag{5}$$

1. Can we learn this matrix as a proxy model?
2. Can we exploit the structure of this matrix?

What is the problem?

What do we **actually** know in practice?

Commonly, we receive measurements of the form [9]

$$v_i \in \mathbb{R}, \quad p_i + jq_i \in \mathbb{C}.$$

1. v_i —voltage ***magnitude***
2. p_i —active (real) power
3. q_i —reactive (imaginary) power[3]

No network model, either!

[3] e.g., from a historical or chosen power factor [9]

What is the problem?

1. How can we recover voltage phasors from their magnitudes?
2. How can we recover a *phaseless model* of the voltage phasors?
3. How can we do this provably?

Contributions

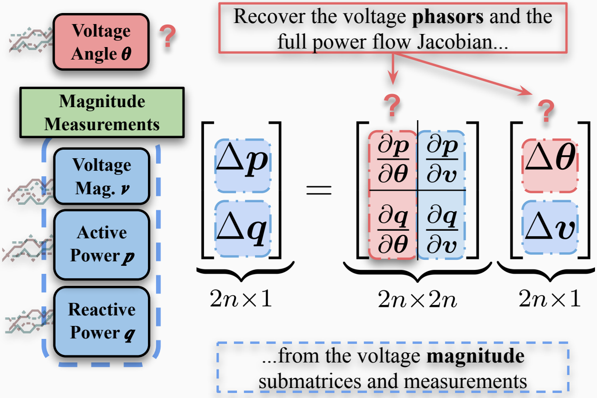

Figure 2: Voltage phasor recovery via power flow Jacobian recovery

How can we solve it?

Classical Phase Retrieval

Given a matrix $A \in \mathbb{C}^{m \times n}$, we want to

$$\text{find } x \in \mathbb{C}^n \quad \text{s. t.} \quad |Ax| = b.$$

where $b \in \mathbb{R}^m$ are real-valued magnitude measurements.

Classical Phase Retrieval

Example applications:

1. X-ray crystallography [5]
2. Electron microscopy [4]
3. Optical and medical imaging [2]
4. Numerous tasks in experimental physics [10, 7]
5. ... Power systems?

Connecting phase retrieval and electric power systems

For any voltage magnitudes $\boldsymbol{v}$, we can represent the **phasor voltages** $\bar{\boldsymbol{v}} \in \mathbb{C}^n$ using a **phase vector** $\boldsymbol{u} \in \mathbb{C}^n$ in the **unit complex ball**

$$|u_i| = 1, \quad \text{atan2}(\text{Im}\{u_i\}, \text{Re}\{u_i\}) = \theta_i, \tag{6}$$

where we require

$$\bar{\boldsymbol{v}} = \text{diag}(\boldsymbol{v})\boldsymbol{u} = \begin{bmatrix} v_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & v_n \end{bmatrix} \boldsymbol{u}. \tag{7}$$

Connecting phase retrieval and electric power systems

Apply the phase retrieval framework to recover the voltage phase

$$\text{find}\, \boldsymbol{\theta} \in (-\pi, \pi]^n \quad \text{st:} \quad \Big| \text{diag}(\boldsymbol{v}) \boldsymbol{u}(\boldsymbol{\theta}) \Big| = \boldsymbol{v}. \tag{8a}$$

Analytical Results

Overview of power flow Jacobian structure

Classical power network physics: well known structural symmetries in the power flow Jacobian matrix.

$$\frac{\partial p_i}{\partial \theta_k} = \begin{cases} v_k \dfrac{\partial q_i}{\partial v_k} & i \neq k, \\ -v_i \dfrac{\partial q_i}{\partial v_i} - 2v_i^2 B_{ii} & i = k. \end{cases} \tag{9a}$$

$$\frac{\partial q_i}{\partial \theta_k} = \begin{cases} -v_k \dfrac{\partial p_i}{\partial v_k} & i \neq k, \\ v_i \dfrac{\partial p_i}{\partial v_i} - 2v_i^2 G_{ii} & i = k \end{cases} \tag{9b}$$

<u>Relates the Power Flow Jacobian Blocks!</u> 🙂

Overview of power flow Jacobian structure

Classical power network physics: well known structural symmetries in the power flow Jacobian matrix [6].

$$\frac{\partial p_i}{\partial \theta_k} = \begin{cases} v_k \dfrac{\partial q_i}{\partial v_k} & i \neq k, \\ -v_i \dfrac{\partial q_i}{\partial v_i} - 2v_i^2 B_{ii} & i = k. \end{cases} \tag{10a}$$

$$\frac{\partial q_i}{\partial \theta_k} = \begin{cases} -v_k \dfrac{\partial p_i}{\partial v_k} & i \neq k, \\ v_i \dfrac{\partial p_i}{\partial v_i} - 2v_i^2 G_{ii} & i = k \end{cases} \tag{10b}$$

Need the network model and parameters 🙁

Solution 1:
A phaseless representation of the Power Flow Jacobian matrix.

New characterization of Power Flow Jacobian structure

Lemma 1: Phaseless power flow Jacobian structure

We can write the partial derivatives of power injections with respect to phase angles without the phase angles and without the grid model.

$$\frac{\partial p_i}{\partial \theta_k} = \begin{cases} v_k \dfrac{\partial q_i}{\partial v_k} & i \neq k \\ v_i \dfrac{\partial q_i}{\partial v_i} - 2q_i & i = k, \end{cases} \tag{11a}$$

$$\frac{\partial q_i}{\partial \theta_k} = \begin{cases} -v_k \dfrac{\partial p_i}{\partial v_k} & i \neq k \\ -v_i \dfrac{\partial p_i}{\partial v_i} + 2p_i & i = k. \end{cases} \tag{11b}$$

These are functions of the **power injection measurements!**

New characterization of the structure of the Power Flow Jacobian

The power-voltage phase angle sensitivity matrices can be expressed as functions $\frac{\partial \boldsymbol{p}}{\partial \boldsymbol{\theta}}, \frac{\partial \boldsymbol{q}}{\partial \boldsymbol{\theta}} : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^{n \times n}$ of the form

$$\frac{\partial \boldsymbol{p}}{\partial \boldsymbol{\theta}}(\boldsymbol{v}, \boldsymbol{q}) = \operatorname{diag}(\boldsymbol{v}) \frac{\partial \boldsymbol{q}}{\partial \boldsymbol{v}} - 2 \operatorname{diag}(\boldsymbol{q}), \tag{12a}$$

$$\frac{\partial \boldsymbol{q}}{\partial \boldsymbol{\theta}}(\boldsymbol{v}, \boldsymbol{p}) = -\operatorname{diag}(\boldsymbol{v}) \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{v}} + 2 \operatorname{diag}(\boldsymbol{p}), \tag{12b}$$

which are implicitly parameterized by $\frac{\partial \boldsymbol{q}}{\partial \boldsymbol{v}}$ and $\frac{\partial \boldsymbol{p}}{\partial \boldsymbol{v}}$.

New characterization of the structure of the Power Flow Jacobian

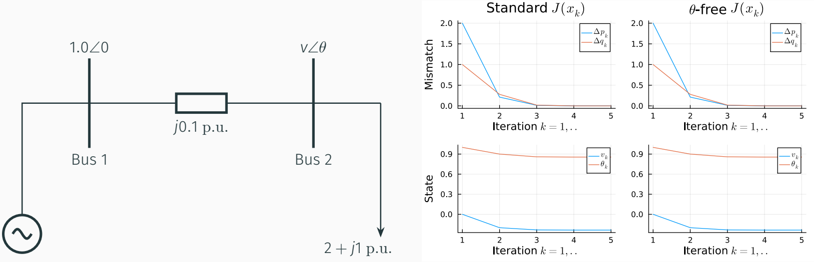

Figure 3: Equivalence of the Newton-Raphson iterations using the θ-free expressions and standard expressions for $\frac{\partial \boldsymbol{p}}{\partial \boldsymbol{\theta}}, \frac{\partial \boldsymbol{q}}{\partial \boldsymbol{\theta}}$ (right) for a simple two bus test case (left).

Solution 2:
Use the power flow Jacobian structure for voltage phase retrieval

Phase retrieval by the power flow Jacobian

Apply a classic phase retrieval algorithm [11] to solve the power flow equations using learned voltage magnitude blocks

$$\underset{\Delta\boldsymbol{\theta}_t,}{\text{minimize}} \quad \left\| \begin{bmatrix} \Delta \boldsymbol{p}_t \\ \Delta \boldsymbol{q}_t \end{bmatrix} - \begin{bmatrix} (?) & \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{v}} \\ (?) & \frac{\partial \boldsymbol{q}}{\partial \boldsymbol{v}} \end{bmatrix} \begin{bmatrix} \Delta \boldsymbol{\theta}_t \\ \Delta \boldsymbol{v}_t \end{bmatrix} \right\|_2^2$$

Learning the Jacobian blocks is well studied [3].

Phase retrieval by the power flow Jacobian

Problem! we do not know $\frac{\partial \boldsymbol{q}}{\partial \boldsymbol{\theta}}, \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{\theta}}$

We have to add them as decision variables.

Phase retrieval by the power flow Jacobian

The phase retrieval program for samples $t = 1, \ldots,$ can then be written as

$$\underset{\Delta\boldsymbol{\theta}_t, \frac{\partial p}{\partial \boldsymbol{\theta}}, \frac{\partial q}{\partial \boldsymbol{\theta}}}{\text{minimize}} \quad \left\| \begin{bmatrix} \Delta p_t \\ \Delta q_t \end{bmatrix} - \begin{bmatrix} \frac{\partial p}{\partial \boldsymbol{\theta}}(v_t, q_t) & \frac{\partial p}{\partial v} \\ \frac{\partial q}{\partial \boldsymbol{\theta}}(v_t, p_t) & \frac{\partial q}{\partial v} \end{bmatrix} \begin{bmatrix} \Delta \boldsymbol{\theta}_t \\ \Delta v_t \end{bmatrix} \right\|_2^2$$

$$\text{subject to:} \quad \text{power flow Jacobian structure!}$$

Solution 3:
Use the power flow Jacobian structure to guarantee voltage phase retrieval

Long story short:

1. Use the Jacobian structure to show when the voltage phase can be uniquely recovered.
2. Do some linear algebra
3. ???
4. Profit!

Take a look at the paper for more details

Guaranteed phase retrieval via spectral theory

Theorem (Phase retrieval from active power injections)

For a set of buses in a network $\mathcal{B} \subset \{1, \ldots, n\}$, if for every bus $i \in \mathcal{B}$, the reactive power differential inequality

$$|q_i| > \frac{1}{2} v_i \left(\sum_{k \in \mathcal{B} \setminus \{i\}} \left| \frac{\partial q_k}{\partial v_i} \right| - \left| \frac{\partial q_i}{\partial v_i} \right| \right) \tag{13a}$$

$$\text{or} \quad |q_i| > \frac{1}{2} \left(\sum_{k \in \mathcal{B} \setminus \{i\}} v_k \left| \frac{\partial q_i}{\partial v_k} \right| - v_i \left| \frac{\partial q_i}{\partial v_i} \right| \right), \tag{13b}$$

holds, then the voltage phase angles can be uniquely recovered from solely the active power (real) injections $\mathbf{p}$.

Guaranteed phase retrieval via spectral theory

Theorem (Phase retrieval from reactive power injections)

Analogously, if for every bus $i \in \mathcal{B}$, the active power differential inequality

$$|p_i| > \frac{1}{2} v_i \left(\sum_{k \in \mathcal{B} \setminus \{i\}} \left| \frac{\partial p_k}{\partial v_i} \right| - \left| \frac{\partial p_i}{\partial v_i} \right| \right) \tag{14a}$$

$$\text{or} \quad |p_i| > \frac{1}{2} \left(\sum_{k \in \mathcal{B} \setminus \{i\}} v_k \left| \frac{\partial p_i}{\partial v_k} \right| - v_i \left| \frac{\partial p_i}{\partial v_i} \right| \right), \tag{14b}$$

holds, then the voltage phase angles can be uniquely recovered from solely the reactive power (imaginary) injections $\boldsymbol{q}$.

Jacobian invertibility guarantees

Side note: can also guarantee Jacobian invertibility

Jacobian invertibility $\iff$ No voltage collapse $\implies$ Phase retrieval

Take a look at the paper for more details

Computational Results

Test case

1. RTS-GMLC network model [1]
2. Open-source
3. Real-world data

How does this compare to classical model-based state estimation?

Comparison with classical state estimation

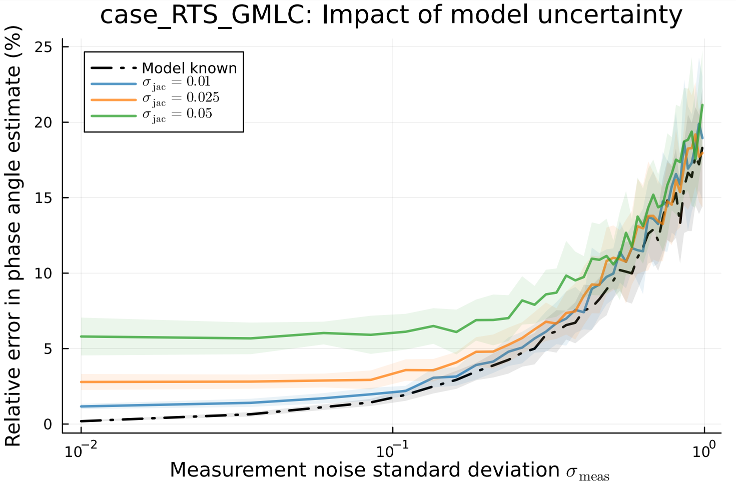

Figure 4: Impact of (*RTS_GMLC*) model uncertainty on recovered phase angle relative error vs. measurement noise level. Shaded regions indicate ± 1 standard deviation of the relative errors computed over 20 bootstraps.

Voltage phasor recovery performance

Voltage phasor recovery performance

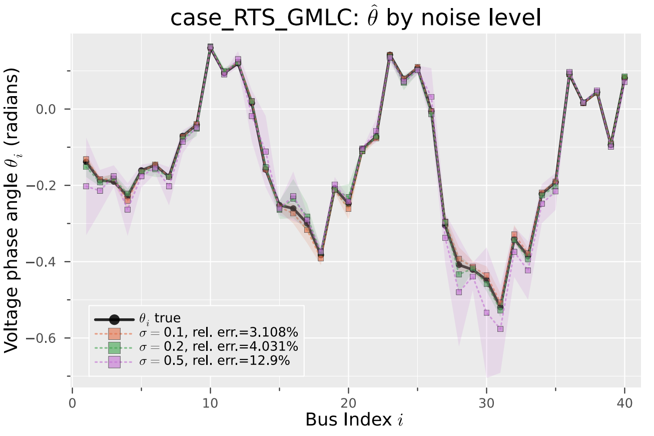

Figure 5: Voltage phasor recovery by measurement noise level.

How about the matrices?

Power flow Jacobian recovery

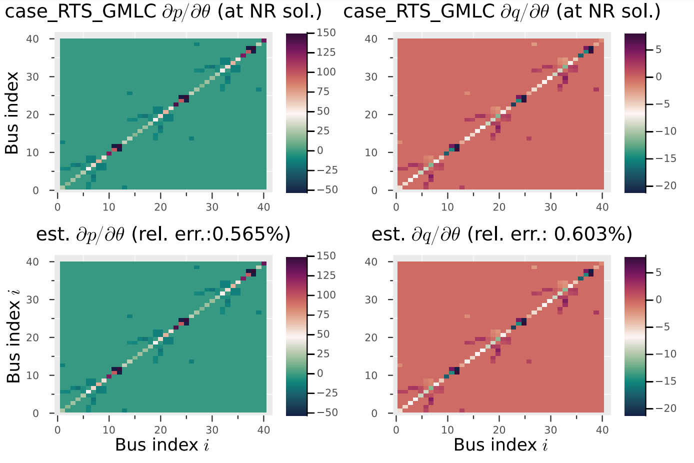

Figure 6: Recovery of the power-phase angle submatrices $\frac{\partial \boldsymbol{p}}{\partial \boldsymbol{\theta}}, \frac{\partial \boldsymbol{q}}{\partial \boldsymbol{\theta}}$ of the power flow Jacobian for the *RTS_GMLC* network via the phase retrieval program.

How about real-time performance?

Real-time voltage phasor recovery performance

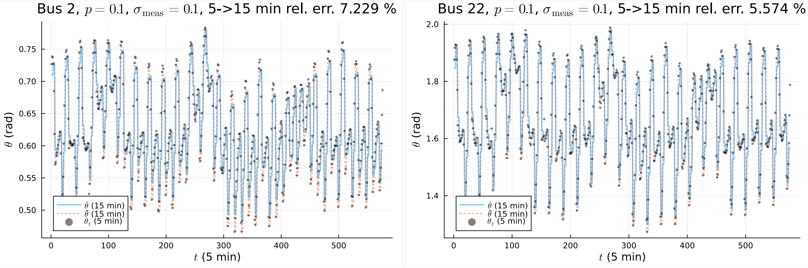

Figure 7: Ground truth (blue) and estimated (orange dashed) voltage phase angles at 15 min. granularity, juxtaposed with ground truth 5 min. granularity voltage phase angles (black dots).

Outlook

What do we have?

1. We can recover voltage phasors from their magnitudes
2. We can recover models of voltage phase angles from their magnitudes
3. This can save a lot of money[4]

[4] A PMU installation costs $40,000-$180,000 each.
"Factors affecting PMU installation costs", US Department of Energy, 2014.

What are the limitations?

1. Bottle necked by measurement frequency (PMUs, milliseconds, other devices, minutes-hours)
2. Bottle necked by measurement type (Reactive power assumptions needed)

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Backup slides

Where are the eigenvalues of a square matrix?

For any matrix $\boldsymbol{A} \in \mathbb{C}^{n \times n}$, by the ***Gershgorin Circle Theorem*** the eigenvalues of $\boldsymbol{A}$ are guaranteed to lie in the union of the $i = 1, \ldots, n$ Gershgorin discs $\mathcal{G}_i(\boldsymbol{A})$ of the matrix, i.e.,

$$\lambda_i(\boldsymbol{A}) \in \bigcup_{i=1}^{n} \mathcal{G}_i(\boldsymbol{A}), \quad i = 1, \ldots, n, \tag{15}$$

where

$$\mathcal{G}_i(\boldsymbol{A}) \triangleq \{w \in \mathbb{C} : |w - A_{ii}| \leq \sum_{k:k \neq i} |A_{ik}|\} \subseteq \mathbb{C}. \tag{16}$$

Classical phase retrieval

Given: $|Ax| = b$, $\boxed{b \in \mathbb{R}^m,}$ what is $x \in \mathbb{C}^n$?

Classical phase retrieval problem:

$$\min_{x\in\mathbb{C}^n, y\in\mathbb{C}^m} ||Ax - y||_2^2 \quad \text{s.t.} \quad |y| = b, \tag{17a}$$

$$\iff \min_{x\in\mathbb{C}^n, u\in\mathbb{C}^m} ||Ax - \operatorname{diag}(b)u|| \quad \text{s.t.} \quad |u| = \mathbb{1}, \tag{17b}$$

$$\iff \min_{u\in\mathbb{C}^m : |u_i|=1\ \forall i\in[\![1,m]\!]} u^* M u, \quad \text{s.t.} \quad M = \operatorname{diag}(b - AA^\dagger) \succ 0. \tag{17c}$$

for any candidate phase u', note that

$$\boxed{\hat{x} = A^\dagger \operatorname{diag}(b)u' = (A^*A)^{-1}A^* \operatorname{diag}(b)u',} \tag{18}$$

Theorem accuracy

Case	# PQ Buses	% Satisfying Thm. 1	r_{worst}
14	9	100.0%	—
24_ieee_rts	13	100.0%	—
ieee30	24	95.83%	1.4×10^{-14}
RTS_GMLC	40	100.0%	—
118	64	100.0%	—
89pegase	77	94.81%	8.32
ACTIVSg200	162	96.91%	0.088
ACTIVSg500	444	94.37%	3.014
ACTIVSg2000	1608	84.83%	28.31

Table 1: Analysis of Theorem 1 for PQ buses of various test cases

Theorem accuracy

Case	# PQ Buses	% Satisfying Thm. **??**	$\sigma_{\max}$
14	9	100.0%	0.876
24_ieee_rts	13	100.0%	0.401
ieee30	24	95.83%	1.437
RTS_GMLC	40	100.0%	0.444
118	64	100.0%	0.473
89pegase	77	100%	0.954
ACTIVSg200	162	100%	0.698
ACTIVSg500	444	99.77%	1.090
ACTIVSg2000	1608	99.69%	1.180

Table 2: Analysis of Theorem **??** for PQ buses of various test cases

Quantity	Value
$\left\lVert \frac{\partial p}{\partial \boldsymbol{\theta}} - \frac{\partial p}{\partial \boldsymbol{\theta}}(v, q) \right\rVert_F \Big/ \left\lVert \frac{\partial p}{\partial \boldsymbol{\theta}} \right\rVert_F$	2.510×10^{-8}
$\left\lVert \frac{\partial q}{\partial \boldsymbol{\theta}} - \frac{\partial q}{\partial \boldsymbol{\theta}}(v, p) \right\rVert_F \Big/ \left\lVert \frac{\partial q}{\partial \boldsymbol{\theta}} \right\rVert_F$	1.725×10^{-7}

Table 3: Verification that the structure expressions of the Lemma hold for multiphase unbalanced networks.